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We propose a twisted parafermion current algebra and its corresponding twisted Z-algebra. We derive the Jacobi-type identities for the twisted parafermion currents and obtain the parafermionic representation of the twisted energy-momentum tensor. As an application, we give a representation of the twisted affine current algebra $A_2^{(2)}$ in terms of the twisted parafermionic currents and a $U(1)$ current.

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The Z_k parafermion algebra was proposed by Zamolodchikov and Fateev [1] in describing a two-dimensional statistical system with Z_k symmetry. The "spin" variables σ_r to each node $r \in L$ on a (square) lattice L take k values ω^q ($q = 0,\ 1\ ,\dots,k-1$), where $\omega = exp(2i\pi/k)$. So it generalizes fermion of the Ising model, which corresponds to the code of Z_2. It is known that there exist various of statistical models, which can be described by the parafermion theory. Examples are the 3-state Potts model ($k = 3$) [1–3] and the Ashkin-Teller model ($k = 4$) [4]. The Z_k parafermion theory contains $k-1$ parafermion currents ψ_l, $l = 1,\ 2,\ \dots,k-1$ and the identity operator $I \equiv \psi_0$. We will write $\psi_l^\dagger = \psi_{k-l} \equiv \psi_{-l}$. Then ψ_l, $\psi_l^\dagger$ obey the following OPEs [1,5–7]:

$$
\begin{aligned}
&\psi_l(z)\psi_{l'}(w)(z-w)^{2ll'/2k} = c_{l,l'}\psi_{l+l'}(w) + \dots, \qquad l+l' < k,\\
&\psi_l(z)\psi_{l'}^\dagger(w)(z-w)^{2l(k-l')/2k} = c_{l,k-l'}\psi_{l-l'}(w) + \dots, \qquad l' < l,\\
&\psi_l(z)\psi_{-l}(w)(z-w)^{2l(k-l)/2k} = 1 + \frac{2d_l}{c_\psi}(z-w)^2 T_\psi(w) + \dots,\\
&T_\psi(z)\psi_l(w) = \frac{d_l}{(z-w)^2}\psi_l(w) + \frac{1}{z-w}\partial_w\psi_l(w) + \dots,
\end{aligned}
\tag{1}
$$

where $d_l = l(k-l)/2k$ is the conformal dimension of ψ_l, $T_\psi(w)$ is the parafermionic energy-momentum tensor and $c_\psi = 2(k-1)/(k+2)$ is the central charge; $c_{ll'}$ are structure constants given in [1,8]

The category for nonlocal operators (parafermions) is the generalized vertex operator algebra [9–11]. The Z_k parafermion algebra was referred to as Z-algebra in [10,11], and the Z_k parafermions are canonically modified Z-algebras acting on certain quotient spaces $A_1^{(1)}$-modules defined by the action of an infinite cyclic group [9–11]. It was proved that the Z-algebra is identical with the $A_1^{(1)}$ parafermion.

Gepner proposed a parafermion algebra associated with any given untwisted affine Lie algebra $\mathcal{G}^{(1)}$ [12]. In this case the central charge of the corresponding Virasoro algebra is $c = \frac{kD}{k+g} - r$, where $D = dim\mathcal{G}$ and $r = rank\mathcal{G}$ are dimension and rank of $\mathcal{G}$, respectively, and g is the dual Coxeter number of $\mathcal{G}^{(1)}$. The OPEs of the untwisted parafermion fields were defined in [12–14]

Parafermion is very useful in string theory [15,16] and $1+1$ dimensional statistical physics [1]. Using the Z_k parafermions and a proper chosen $U(1)$ current, one can represent the $A_1^{(1)}$ affine current algebra [1] and the $N = 2$ superconformal algebra [15,16], which is important in the study of the mirror symmetry [17]. The Z_k parafermion characters and their singular vectors were studied in [18]. In a very recent work by Maldacena, Moore and Seiberg, the D-branes were constructed with the help of the Z_k parafermions [19].

In this paper, we will propose a twisted parafermion algebra. We derive the corresponding twisted Z-algebra and the Jacobi-type identities for the twisted parafermion currents. Moreover, we obtain the parafermionic energy-momentum tensor and a parafermionic representation of the twisted affine current algebra $A_2^{(2)}$.

It is well-known that Euclidian correlation functions are defined only if operators in the correlators are time-ordered [20]. In the radial picture, $|z| > |w|$ means that z is later than w. In the Euclidian functional integral

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definition of correlation functions, the time ordering is automatic. So, in Euclidian field theory the operator products $A(z)B(w)$ are only defined for $|z| > |w|$. Therefore the radial ordering is implied throughout this paper.

Now we propose the twisted parafermion current algebra:

$$\begin{aligned}
\psi_l(z)\psi_{l'}(w)(z-w)^{ll'/2k} &= \frac{\delta_{l+l',0}}{(z-w)^2} + \frac{\varepsilon_{l,l'}}{z-w}\psi_{l+l'}(w) + \cdots, \\
\psi_{\tilde{l}}(z)\psi_{\tilde{l}'}(w)(z-w)^{ll'/2k} &= \frac{\delta_{\tilde{l}+\tilde{l}',0}}{(z-w)^2} + \frac{\varepsilon_{\tilde{l},\tilde{l}'}}{z-w}\psi_{l+l'}(w) + \cdots, \\
\psi_l(z)\psi_{\tilde{l}'}(w)(z-w)^{ll'/2k} &= \frac{\varepsilon_{l,l'}}{z-w}\psi_{\tilde{l}+\tilde{l}'}(w) + \cdots,
\end{aligned} \tag{2}$$

where $l,\ l' = \pm 1$ and $\tilde{l},\ \tilde{l}' = \tilde{0}, \pm\tilde{1}, \pm\tilde{2}$; $\varepsilon_{l,l'}$, $\varepsilon_{\tilde{l},\tilde{l}'}$ and $\varepsilon_{l,\tilde{l}'}$ are structure constants. If we denote ψ_l or $\psi_{\tilde{l}}$ by Ψ_a, then we can rewrite the above relations as:

$$\Psi_a(z)\Psi_b(w)(z-w)^{ab/2k} = \frac{\delta_{a+b,0}}{(z-w)^2} + \frac{\varepsilon_{a,b}}{z-w}\Psi_{a+b}(w) + \sum_{n=0}^{\infty}(z-w)^n[\Psi_a\Psi_b]_{-n}, \equiv \sum_{n=-2}^{\infty}(z-w)^n[\Psi_a\Psi_b]_{-n}, \tag{3}$$

where $a, b = \tilde{0}, \pm 1, \pm\tilde{1}, \pm\tilde{2}$. So we have $[\Psi_a\Psi_b]_l = 0$ $(l \geq 3)$, $[\Psi_a\Psi_b]_2 = \delta_{a+b,0}$ and $[\Psi_a\Psi_b]_1 = \varepsilon_{a,b}\Psi_{a+b}$. For consistency, $\varepsilon_{a,b}$ must have the properties: $\varepsilon_{a,b} = -\varepsilon_{b,a} = -\varepsilon_{-a,-b} = \varepsilon_{-a,a+b}$ and $\varepsilon_{a,-a} = 0$. Due to the mutually semilocal property between two parafermions, the radial ordering products are multivalued functions. So we define the radial ordering product of (generating) twisted parafermions(TPFs) as

$$\Psi_a(z)\Psi_b(w)(z-w)^{ab/2k} = \Psi_b(w)\Psi_a(z)(w-z)^{ab/2k}, \tag{4}$$

which, like the untwisted case, is an extension of the fermion (i.e. $ab = 1,\ k = 1$) and boson (i.e. $k \to \infty$).

For every field in the parafermion theory there are a pair of charges $(\lambda, \bar{\lambda})$, which take values in the weight lattice. We denote such a field by $\phi_{\lambda,\bar{\lambda}}(z,\bar{z})$ [1,12,13]. The OPE of ψ_a with $\phi_{\lambda,\bar{\lambda}}(z,\bar{z})$ is given by

$$\psi_a(z)\phi_{\lambda,\bar{\lambda}}(w,\bar{w}) = \sum_{m=-\infty}^{\infty}(z-w)^{-m-1-a\lambda/2k}A_m^{a,\lambda}\phi_{\lambda,\bar{\lambda}}(w,\bar{w}), \tag{5}$$

where $m \in Z$ (Ramond sector) for $a = l$ and $m \in Z + \frac{1}{2}$ (Neveu-Schwarzsector) for $a = \tilde{l}$. The action of $A_m^{a,\lambda}$ on $\phi_{\lambda,\bar{\lambda}}(z)$ is defined by the integration

$$A_m^{a,\lambda}\phi_{\lambda,\bar{\lambda}}(w,\bar{w}) = \oint_{c_w} dz\ (z-w)^{m+a\lambda/2k}\psi_a(z)\phi_{\lambda,\bar{\lambda}}(w,\bar{w}), \tag{6}$$

where c_w is a contour around w. Consider the difference of integrals

$$\oint\oint du\ dz \Psi_a(u)\Psi_b(z)\phi_{\lambda,\bar{\lambda}}(w,\bar{w})(u-z)^{-p-1+ab/2k} \times (u-w)^{m+q+1+a\lambda/2k}(z-w)^{n+p-q+b\lambda/2k}, \tag{7}$$

along two contours satisfying $|u-w| > |z-w|$ and $|u-w| < |z-w|$, respectively. The difference of the two contour integrals can be expressed as a two-fold integration of an u-contour enclosing z once followed by an z-contour enclosing w once. Properly carrying out the Taylor expansion of $(u-z)^x$, we then obtain the so-called twisted Z-algebra relations,

$$\begin{aligned}
&\sum_{l=0}^{\infty} C^{(l)}_{-p-1+ab/2k}\left[A^{a,\lambda+b}_{m-l-p+q}A^{b,\lambda}_{n+l+p-q} + (-1)^p A^{b,\lambda+a}_{n-l-q-1}A^{a,\lambda}_{m+l+q+1}\right] \\
&= C^{(p+2)}_{m+q+1+a\lambda/2k}\delta_{a,-b}\delta_{m,-n} + C^{(p+1)}_{m+q+1+ab/2k}\varepsilon_{a,b}A^{a+b,\lambda}_{m+n} + \sum_{r=0}^{\infty} C^{p-r}_{m+q+1+ab/2k}[\psi_a\psi_b]^{\lambda}_{-r,m+n},
\end{aligned} \tag{8}$$

where $p = 2q$ or $2q + 1$ and $[\Psi_a\Psi_b]^{\lambda}_{-m,n}$ is defined by $[\Psi_a\Psi_b]^{\lambda}_{-m,n}\phi_{\lambda,\bar{\lambda}}(w,\bar{w}) = \oint_{c_w} dz\ (z-w)^{m+n+1+(a+b)\lambda/2k}[\Psi_a\Psi_b]_{-m}(z)\phi_{\lambda,\bar{\lambda}}(w,\bar{w})$.

Let A_a and B_b be two arbitrary functions of the twisted parafermions with charges a and b, respectively. These fields are local (a or $b = 0$) or semilocal (a and $b \neq 0$). The OPEs can be written as

$$A_a(z)B_b(w)(z-w)^{ab/2k} = \sum_{n=-[h_A+h_B]}^{\infty} [A_aB_b]_{-n}(w)(z-w)^n, \tag{9}$$

where $[h_A]$ stands for the integral part of the dimension of A. Hence we have $[A_aB_b]_n(w) = \oint_w dz\ A_a(z)B_b(w)(z-w)^{n-1+ab/2k}$. It is easy to find the following relation between the three-fold radial ordering products

$$\begin{aligned}&\left\{\oint_w du \oint_w dz\ R(A(u)R(B(z)C(w))) - \oint_w dz \oint_w du\ (-)^{ab/2k} R(B(z)R(A(u)C(w)))\right.\\ &\left.- \oint_w dz \oint_z du\ R(R(A(u)B(z))C(w))\right\}(z-w)^{p-1+bc/2k}(u-w)^{q-1+ac/2k}(u-z)^{r-1+ab/2k} = 0,\end{aligned} \tag{10}$$

where integers $p,\ q,\ r$ are in the regions: $-\infty < p \le [h_B+h_C]$, $-\infty < q \le [h_C+h_A]$, $-\infty < r \le [h_A+h_B]$; and $a,\ b,\ c$ are parafermionic charges of the fields A, B and C, respectively. The above equation is an extension of $A(BC) - B(AC) - [A,B]C = 0$. Performing the binomial expansion, we obtain the following twisted Jacobi-like identities:

$$\begin{aligned}&\sum_{i=p}^{[h_B+h_C]} C^{(i-p)}_{r-1+ab/2k}[A[BC]_i]_{Q-i}(w) + (-)^r \sum_{j=q}^{[h_C+h_A]} C^{(j-q)}_{r-1+ab/2k}[B[AC]_j]_{Q-j}(w)\\ &= \sum_{l=r}^{[h_B+h_A]} (-)^{(l-r)} C^{(l-r)}_{q-1+ac/2k}[[AB]_lC]_{Q-l}(w),\end{aligned} \tag{11}$$

where $Q = p+q+r-1, C^{(l)}_x = \frac{(-)^l x(x-1)\ldots(x-l+1)}{l!}$ and $C^{(0)}_0 = C^{(0)}_n = C^{(l)}_{-1} = 1,\ C^{(l)}_p = 0$, for $l > p > 0$. This identity will be used extensively for our purpose. Performing analytic continuation one obtains

$$[BA]_r(w) = \sum_{t=r}^{[h_A+h_B]} \frac{(-)^t}{(t-r)!}\partial^{t-r}[AB]_t(w). \tag{12}$$

We remark that $A,\ B,\ C$ can be any composite operators and we can calculate any coefficient in the OPEs from the fundamental equation (3).

For the twisted parafermion theory to be a conformal field theory, we require that the spin-2 terms in the OPEs contain a energy-momentum tensor. It is obvious that the spin-2 terms in the OPE are $[\Psi_a\Psi_b]_0$. Since the parafermionic charge for the energy-momentum tensor should be zero, so the relevant terms are $[\Psi_a\Psi_{-a}]_0$. Note that $[\Psi_a\Psi_{-a}]_0(z) = [\Psi_{-a}\Psi_a]_0(z)$, we calculate the OPE of $[\Psi_a\Psi_{-a}]_0$ with Ψ_a and $[\Psi_b\Psi_{-b}]_0$. Setting $Q = p = 2,\ q = 1$ and $r = 0$ in (11), we have

$$[[\Psi_a\Psi_{-a}]_0\Psi_b]_2 = \delta_{a,b}\psi_a + \varepsilon_{a,b}\varepsilon_{-a,a+b}\Psi_b + (1+a^2/2k)\delta_{a,-b}\Psi_{-a} + \frac{ab}{4k}(1-\frac{ab}{2k})\Psi_b. \tag{13}$$

From the general theory of conformal field theory [21], the conformal dimension of Ψ_a should be $1-\frac{a^2}{4k}$. So we impose the constraints:

$$\sum_a \varepsilon_{a,b}\varepsilon_{-a,a+b} = \frac{6-b^2}{k}, \qquad \sum_a ab = 0, \qquad \sum_a (ab)^2 = 12b^2. \tag{14}$$

One solution to these constraints is given by $\varepsilon_{\tilde{1},-\tilde{2}} = \varepsilon_{1,\tilde{1}} = \varepsilon_{-1,\tilde{2}} = \frac{1}{\sqrt{k}}$ and $\varepsilon_{1,-\tilde{1}} = \varepsilon_{\tilde{0},\tilde{1}} = \sqrt{\frac{3}{2k}}$. Then we have $\sum_a [[\Psi_a\Psi_{-a}]_0\Psi_b]_2 = \frac{2k+6}{k}\left(1-\frac{b^2}{4k}\right)\Psi_b$. Choose a proper normalization and write $T_\psi = \frac{k}{2k+6}\sum_a[\Psi_a\Psi_{-a}]_0$. Then $[T_\psi\Psi_b]_2 = \left(1-\frac{b^2}{2k}\right)\Psi_b$. Repeating the above process, we obtain

$$[[\Psi_a\Psi_{-a}]_0\Psi_b]_1 = \frac{1}{2}\varepsilon_{-a,b}\varepsilon_{a,-a+b}\partial\Psi_b + \frac{1}{2}\varepsilon_{a,b}\varepsilon_{-a,a+b}\partial\Psi_b + (1+a^2/2k)\delta_{a,b}\partial\Psi_a + (1+a^2/2k)\delta_{-a,b}\partial\Psi_{-a} \tag{15}$$

or equivalently $[T_\psi\Psi_b]_1 = \partial\Psi_b$. These the above results can be written in the form of OPEs

$$T_\psi(z)\Psi_b(w) = \frac{1-b^2/4k}{(z-w)^2}\Psi_b(w) + \frac{1}{z-w}\partial\Psi_b(w) + \ldots. \tag{16}$$

It follows that the conformal dimension of the twisted parafermion is 1, $1-\frac{1}{4k}$ or $1-\frac{1}{k}$, for a given level k. Carrying out a similar program for T, we obtain the OPE:

$$T_\psi(z)T_\psi(w)=\frac{c_{\rm tpf}/2}{(z-w)^4}+\frac{2T_\psi(w)}{(z-w)^2}+\frac{\partial T_\psi(w)}{z-w}+\ldots, \tag{17}$$

where $c_{\rm tpf}=7-\frac{24}{k+3}=\frac{8k}{k+3}-1$ is the central charge of the twisted parafermion theory.

One of the applications of the twisted parafermionic currents is that they give a representation of the twisted affine current algebra $A_2^{(2)}$. Introduce eight currents:

$$\begin{aligned}
&j^+(z)=2\sqrt{k}\psi_1(z)e^{\frac{i}{\sqrt{2k}}\phi_0(z)}, && j^-(z)=2\sqrt{k}\psi_{-1}(z)e^{-\frac{i}{\sqrt{2k}}\phi_0(z)},\\
&j^0(z)=2\sqrt{2k}i\partial\phi_0(z), && J^+(z)=2\sqrt{k}\psi_{\tilde{1}}(z)e^{\frac{i}{\sqrt{2k}}\phi_0(z)},\\
&J^-(z)=2\sqrt{k}\psi_{-\tilde{1}}(z)e^{-\frac{i}{\sqrt{2k}}\phi_0(z)}, && J^{++}(z)=2\sqrt{k}\psi_{\tilde{2}}(z)e^{i\sqrt{\frac{2}{k}}\phi_0(z)},\\
&J^{--}(z)=2\sqrt{k}\psi_{-\tilde{2}}(z)e^{-i\sqrt{\frac{2}{k}}\phi_0(z)}, && J^0(z)=2\sqrt{6k}\psi_{\tilde{0}}(z).
\end{aligned}$$

where ϕ_0 is an $U(1)$ current obeying $\phi_0(z)\phi_0(w)=-ln(z-w)$. Then it can be checked that the above currents satisfy the OPEs of the twisted affine currents algebra $A_2^{(2)}$ [22].

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[1] A.B.Zamolodchikov and V.A.Fateev, Sov. Phys, JETP **62**, 215 (1985).
[2] V.A.Fateev and A.B.Zamolodchikov, Nucl.Phys.**B280**, 644 (1987).
[3] VI.S. Dotsenko. Nucl. phys.**B 235**, 54 (1984).
[4] S.K. Yang, Nucl. Phys.**B285**, 639 (1987).
[5] D. Nemeschansky, Phys. Lett.**B 224**, 121 (1989).
[6] D.Gepner and Z.Qiu, Nucl. Phys.**B285**, 423 (1987).
[7] J. Distler and Z. Qiu, Nucl. Phys.**B336**, 533 (1990).
[8] VI.S. Dotsenko and V.A. Fateev, Nucl. Phys. **B240**, 312 (1984); **B251**, 691 (1985).
[9] C.Y. Dong and J. Lepowsky, *Generalized Vertex Algebras and Relative Vertex Operators*, Progress in Math. **112**, Birkhauser, 1993.
[10] J. Lepowsky and M. Primc, *Structure of the standard $A_1^{(1)}$-modules in the homogeneous picture*, in: *Vertex Operators in Mathematics and Physics, Proc. 1983 M.S.R.I. conference*, ed. by J. Lepowsky, S. Mandelstam and I. Singer, springer-Verlag, New York, 1985, 143-162.
[11] J. Lepowsky and R. Wilson, *comm. Math. Phys.* **62** , (1978), 43-53; *Invent. Math.* **77**, (1984), 199-200; *Invent. Math.* **79**, (1985), 417-442; *Z-algebras and the Rogers-Ramanujan identities*, in: *Vertex Operators in Mathematics and Physics, Proc. 1983 M.S.R.I. conference*, ed. by J. Lepowsky, S. Mandelstam and I. Singer, springer-Verlag, New York, 1985, 97-142.
[12] D. Gepner, Nucl. Phys. **B290**, 10(1987).
[13] X.M. Ding, H. Fan, K.J. Shi, P. Wang and C.Y. Zhu, Phys. Rev. Lett. **70**, 2228 (1993); Nucl. Phys.**B422**, 307 (1994).
[14] P. Wang and X.M. Ding, Phys. Lett.**B335**, 56 (1994).
[15] Z.Qiu. Phys. Lett.**B 198**, 497 (1987).
[16] M.Freeman and P.West, Phys. Lett.**B314**, 320 (1993).
[17] *Essays on mirror manifolds* , ed. by S.T. Yau, Hong Kong, International Press, 1992.
[18] P. Jacob and P. Mathieu, Nucl. Phys. **B587**, 514 (2000); hep-th/0006233.
[19] J. Maldacena, G. Moore and N. Seiberg, *JHEP***0107**, 046 (2001). hep-th/0105038.
[20] C. Itzykson and J.-B. Zuber, *Quantum Field Theory*, McGraw-Hill Inc. 1980.
[21] A.A. Belavin, A.M. Polyakov, A.B. Zamolodchikov, *Nucl. Phys.* **B241**, 333(1984).
[22] X.M. Ding, M.D. Gould and Y.Z. Zhang, *Twisted $sl(3)_k^{(2)}$ Current Algebra: Free Field Representation and Screening Currents*, hep-th/0109009, to be published.